The Identification Power of Equilibrium in Games: The Supermodular Case*

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September 2007

Abstract

This paper discusses how the analysis of Aradillas-Lopez and Tamer (2007) on the identification power of equilibrium in games can be extended to supermodular games. These games embody models that exhibit strategic complementarity, an important and empirically relevant class of economic models. In these games, the extreme points of the Nash Equilibrium and rationalizable strategy sets coincide, as shown by Milgrom and Roberts (1990) and Vives (1990). We discuss how this result facilitates a comparative analysis of the relative identification power of equilibrium and weaker notions of rational behavior. As an illustrative example, we consider a differentiated product oligopoly pricing game in which firms' prices are strategic complements.

JEL classification: C31, L13.

Keywords: identification, Nash Equilibrium, rationalizability, differentiated products, oligopoly

*We thank Arie Beresteanu and participants at the 2007 Joint Statistical Meetings session on "Inference in Simple Dynamic Games with Multiple Equilibria" for comments and discussion. We also thank the Editor, Serena Ng, and a reviewer for their comments. Molinari gratefully acknowledges financial support from the National Science Foundation grant SES-0617482. Rosen gratefully acknowledges financial support from the Economic and Social Research Council through the ESRC Centre for Microdata Methods and Practice grant RES-589-28-0001. All errors are our own.

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1 Introduction

To date, Nash Equilibrium has been the primary solution concept employed in econometric models of games. But the theory of games need not rely on the use of Nash Equilibrium, nor any of its refinements, as a solution concept. Indeed, the seminal work of von Neumann and Morgenstern (1944) preceded the introduction of Nash equilibrium (Nash (1950)). In their later papers that introduce rationalizability to the literature, Bernheim (1984) and Pearce (1984) show that common knowledge of rationality yields a different solution concept than Nash equilibrium, rationalizability, and they argue thus that in some settings deductive reasoning may not be sufficient to imply Nash behavior.

This point clearly has implications for econometricians who employ game theoretic models to infer agents' payoffs, behavior, and strategic interactions. In settings where it is not clear how players settle on Nash equilibria, and in particular when there are multiple equilibria, it would therefore seem desirable to deduce what inferences can be made robust to the assumption of Nash equilibrium. For this reason, we see the study of the identification power of equilibrium undertaken by Aradillas-Lopez and Tamer (2007, henceforth ALT) as an important step in the econometrics literature. The notion of rationalizability seems a natural one to adopt, in that it is a deductive implication of common knowledge of rationality.

That said, the task of drawing inferences in econometric models of games based only on rationalizability or only finite levels of rationality rather than equilibrium is a difficult one. Even when equilibrium is employed as a solution concept, multiple equilibria are known to substantially complicate identification and inference. Issues raised by multiple equilibria would only seem to be exacerbated by weakening the Nash equilibrium assumption. Given these considerations, we see the constructive identification results provided by ALT in the games they study as positive findings.

As we elaborate here, the results derived also relate to the literature on supermodular games. The applicability of supermodular games or similar models that exhibit strategic complementarity has been demonstrated in many economic contexts, including models of search (Diamond (1982)), technology adoption (Farrell and Saloner (1986), Katz and Shapiro (1986)), R&D competition

(Reinganum (1981)), oil drilling (Katz and Shapiro (1986)), and firm production (Beresteanu (2005)), as well as additional references given by Milgrom and Roberts (1990), Vives (1999), and Vives (2005). Such games have additional structure, often commensurate with economic theory, that can be exploited in conjunction with the analysis of both Nash equilibrium and rationalizable strategies as shown by Milgrom and Roberts (1990). In supermodular games, Milgrom and Roberts (1990) show that the sets of pure strategy Nash Equilibria and rationalizable strategies have identical lower and upper bounds. Moreover, the equilibria of supermodular games exhibit monotone comparative statics (Milgrom and Shannon (1994)), for which Echenique and Komunjer (2006) develop a formal test that is robust to multiple equilibria. Though the games considered by ALT are not supermodular, we show below that a transformed version of their entry games exhibits supermodularity, and that this can be exploited to apply Milgrom and Roberts' results. Then we discuss how these results can be used to study the identification power of Nash equilibrium relative to rationalizability or lower levels of rationality in supermodular games more generally, with a differentiated product oligopoly pricing game as our leading illustrative example.

2 The Entry Model

Consider the complete information entry game of ALT, but redefine the strategy space as $s_1 = a_1$ and $s_2 = -a_2$. Thus, $s_1 = 1$ if firm 1 is present in the market, and $s_1 = 0$ otherwise, while $s_2 = -1$ if firm 2 is in the market, and $s_2 = 0$ otherwise. This simple reformulation of the usual entry game strategy space amounts only to a relabeling of strategies without modifying the game's economic content, and is based on the reformulation used by Vives (1999, Chapter 2.2.3) in the analysis of Cournot duopoly. Given this reformulation of player 2's strategy, the firms' profit functions are given by

$$\begin{aligned}
\pi_1 &= s_1 \left(t_1 - \alpha_1 s_2 \right), \\
\pi_2 &= -s_2 \left(t_2 + \alpha_2 s_1 \right),
\end{aligned}$$

where t_1 and t_2 are, as in ALT, each players' type. The redefined game is supermodular because (i) players' strategy sets are complete lattices, (ii) the profit functions are order continuous and bounded from above, (iii) each profit function π_j is supermodular in s_j for fixed s_{-j}, and (iv) each π_j has increasing differences in s_j and s_{-j}. These conditions are sufficient for the game to be supermodular, see e.g. conditions (A1) - (A4) of Milgrom and Roberts (1990), and this guarantees that each firm's best response functions are increasing in their rival's strategy. Unfortunately, this reformulation cannot be used to obtain a supermodular representation of the entry game (or the Cournot game) when there are more than 2 players and entry decisions are strategic substitutes, i.e. the α's are negative. When instead entry decisions are complementary, as in Borzekowski and Cohen (2005) for example, then the game is supermodular without reparameterization.

Now consider the *incomplete* information entry game of ALT, with the redefined action space given above. The result that the transformed game is supermodular carries through here as well. We follow along the lines of Vives (1999, Chapter 2.7.3) to justify this result, adopting similar notation in the context of this Bayesian game. Given the vector of parameters $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, let $\pi_i(\mathbf{s}, \mathbf{t}; \boldsymbol{\alpha})$ be the ex-post payoff of player i when the vector of actions are $\mathbf{s} = (s_1, s_2)$ and the realized types are $\mathbf{t} = (t_1, t_2)$. Action spaces, profit functions, type sets and the prior distribution of types are all common knowledge. A pure strategy for player i is a measurable function that assigns an action to every possible type of the player. Let Σ_i be the strategy space of player i and identify strategies σ_i and τ_i if they are equal with probability 1. Let $\Pi_i(\sigma_1, \sigma_2) = E[\pi_i(\sigma_1(t_1), \sigma_2(t_2), \mathbf{t}; \boldsymbol{\alpha})]$ be the expected payoff to player i when player $-i$ uses strategy σ_{-i}, and denote by $\beta_i : \Sigma_{-i} \rightarrow \Sigma_i$ player i's best response correspondence. Then a Bayesian Nash equilibrium is a strategy profile $\boldsymbol{\sigma}$ such that $\sigma_i \in \beta_i(\sigma_{-i})$ for $i = 1, 2$. Vives (1999, Chapter 2.7.3) defines a natural order in the strategy space as $\sigma_i \leq \sigma'_i$ if $\sigma_i(t_i) \leq \sigma'_i(t_i)$ component-wise with probability 1. It then follows that the strategy spaces in the Bayesian game are complete lattices, and that $\Pi_i(\sigma_1, \sigma_2)$ is supermodular in σ_i and has increasing differences in (σ_i, σ_{-i}), because these properties of $\pi_i(\cdot, \cdot)$ are preserved under integration.

This shows that once the strategy space is reformulated, both the complete and incomplete information entry games are supermodular. Hence, we can apply Milgrom and Roberts' Theorem

5 (or equivalently Vives' (1990) Theorem 6.1), which we repeat here in the notation of the present game for clarity.

Theorem 5 (Milgrom and Roberts 1990). *Let* Γ *be a supermodular game. For each player* j*, there exist largest and smallest serially undominated strategies,* $\bar{s}_j$ *and* $\underline{s}_j$*. Moreover, the strategy profiles* $(\underline{s}_1, \underline{s}_2)$ *and* $(\bar{s}_1, \bar{s}_2)$ *are pure Nash equilibrium profiles.*

This theorem implies that in supermodular games both Nash equilibrium and rationalizability yield identical lower and upper bounds on observable strategy profiles. This has two consequences. First, when the Nash equilibrium is unique, it gives the *only* rationalizable strategy profile. In this case, the game is dominance solvable, as iterated elimination of dominated strategies picks out the unique Nash Equilibrium. Second, when there are multiple equilibria, the largest and smallest equilibrium strategy profiles are also the largest and smallest rationalizable strategy profiles. In the context of the complete information entry game, this means that in the region of multiplicity where $(-t_1/\alpha_1, -t_2/\alpha_2) \in [0,1]^2$, the equilibrium profiles $(s_1, s_2) = (0, -1)$ and $(s_1, s_2) = (1, 0)$ are both rationalizable, and are also the smallest and largest such profiles. On the other hand, $(0, 0)$ and $(1, -1)$ are also rationalizable outcomes, but they are not pure strategy equilibria. They may, however, result from mixed strategy equilibrium, yielding the conclusion that in the region of multiplicity, any outcome is observable when firms play either Nash Equilibrium or rationalizable strategies. For the game of complete information, these implications coincide with the analysis of ALT illustrated in their Figure 2.

The theorem of Milgrom and Roberts applies to all supermodular games. It does not, however, provide a characterization of feasible outcomes under k-level rationality for finite k. Thus, it is not sufficient to provide a characterization of the identifiable features of the model with k*-level* rationality, $k < \infty$. Rather, one must use iteration of levels of rationality to determine the set of feasible outcomes under this restriction, as done by ALT. Clearly the model predictions under these two different solution concepts may differ even for $k \to \infty$, unless the game has a unique Nash equilibrium. Still, the property that the k-level largest and smallest rational strategies shrink to the largest and smallest Nash equilibria as $k \to \infty$ means that imposing k-level rationality should have some identifying power for some finite k, at least in those games where equilibrium restrictions do. In contrast, in games that are not supermodular, it is possible that the bounds on rationalizable outcomes are much wider than those implied by equilibrium behavior. If this is the case, then imposing k-level rationality may not have much identification power. For this reason, we believe that the study of the relative identifying power of equilibrium, rationalizability, and k-level rationality in supermodular games in the same vein as ALT is rather promising.

3 A Differentiated Product Pricing Game

To illustrate the applicability of this type of analysis in the context of supermodular games, we consider a simple linear duopoly model of price competition in the sale of differentiated products, though similar analysis can also be applied to non-linear models with more than two firms. There are two firms denoted $j = 1, 2$ that each produce their own version of a differentiated product in a given market. The firms set their prices simultaneously, and consumers in the market then choose how much of each product to buy. Market demand is assumed to be given by the following linear specification for the differentiated products:

$$Q_1(\mathbf{p}) = \alpha_1 - \beta_{11}p_1 + \beta_{12}p_2 + u_1, \tag{1}$$

$$Q_2(\mathbf{p}) = \alpha_2 - \beta_{22}p_2 + \beta_{21}p_1 + u_2, \tag{2}$$

where p_j denotes the price set for product j, and $Q_j(\mathbf{p})$ denotes the quantity demanded for product j given prices $\mathbf{p} \equiv (p_1, p_2)'$. We assume that the differentiated products are substitutes, and thus impose the following assumption:

Assumption A0: $\beta_{11} > \beta_{12} > 0$, $\beta_{22} > \beta_{21} > 0$.

Assumption A0 guarantees that each product's demand is downward sloping in its own price and increasing in its competitors price. It additionally stipulates that demand for each product is more sensitive (in absolute terms) to price changes in its own price than price changes of the competing product. A0 also guarantees that the dominant diagonal condition is satisfied, which is sufficient for uniqueness of the Nash Equilibrium of the pricing game.

Assume that each firm has marginal costs given by

$$mc_j = \gamma_{j0} + \gamma_{j1} w_1 + \gamma_{j2} w_2 + \epsilon_j \equiv \mathbf{w}\boldsymbol{\gamma}_j + \epsilon_j, \tag{3}$$

where w_1, w_2 are marginal cost shifters, such as factor prices, observable to both the econometrician and the firms, and $\mathbf{w} = [1\ w_1\ w_2]$ and $\boldsymbol{\gamma}_j = \left[\gamma_{j0}\ \gamma_{j1}\ \gamma_{j2}\right]'$. (ϵ_1, ϵ_2) are additional firm-specific cost shifters which are assumed to be perfectly observed by the firms, but unobserved by the econometrician. We have assumed that firms have the same observable cost shifters, but this is not crucial to the analysis.

Each firm seeks to maximize their profits:

$$\begin{aligned} \pi_1(\mathbf{p}) &= (p_1 - mc_1)\, Q_1(\mathbf{p}), \\ \pi_2(\mathbf{p}) &= (p_2 - mc_2)\, Q_2(\mathbf{p}). \end{aligned}$$

Prices are strategic complements, as each firm's profits are increasing in its rival's price. Using the associated first order conditions, the firms' reaction functions are given by

$$p_1^*(\tilde{p}_2) = \frac{\alpha_1 + \beta_{12}\tilde{p}_2 + u_1 + \beta_{11} mc_1}{2\beta_{11}}, \tag{4}$$

$$p_2^*(\tilde{p}_1) = \frac{\alpha_2 + \beta_{21}\tilde{p}_1 + u_2 + \beta_{22} mc_2}{2\beta_{22}}, \tag{5}$$

where $\tilde{p}_2$ $(\tilde{p}_1)$ is firm 1's (firm 2's) conjecture for p_2 (p_1). In Nash equilibrium firms correctly anticipate their rivals' equilibrium prices, which delivers the unique vector (p_1, p_2) such that $p_1 = p_1^*(p_2)$ and $p_2 = p_2^*(p_1)$. If, on the other hand, firms are only k-level rational, then they need not correctly anticipate their opponents' strategies. Rather, they will each play a best-reply to some $k-1$ level rational strategy of their opponent.

Under Assumption A0, Theorem 5 of Milgrom and Roberts implies immediately that if the firms play rationalizable strategies (i.e. they are k-rational for all k), the only rationalizable outcome is the unique Nash equilibrium. If instead one assumes that each firm is only k-level rational for some fixed k, then iterated elimination of dominated strategies can be used in conjunction with (4) and (5) to deliver the set of (p_1, p_2) outcomes consistent with k-level rationality for any parameter vector and realization of unobservables. In similar fashion to the analysis of ALT, this can then be used to formulate a consistent set estimate for model parameters, which can in turn be compared to estimates based on a Nash equilibrium assumption. We illustrate this below. Before proceeding, we shall make explicit six additional assumptions that will be imposed throughout:

Assumption A1: The econometrician observes a random sample of observations of (w_1, w_2, p_1, p_2) from a large cross section of markets drawn from a population distribution that satisfies (1), (2), (3), as well as assumptions A0 and A2-A6 below.

Assumption A2: $E[\epsilon_j] = E[\epsilon_j \cdot w_l] = E[u_j \cdot w_l] = E[u_j] = 0$, $\forall j \in \{1, 2\}$, $\forall l \in \{1, 2\}$

Assumption A3: $E\left[(1, w_1, w_2)' \cdot (1, p_1, p_2)\right]$ has full rank.

Assumption A4: $E\left[(1, w_1, w_2)' \cdot (1, w_1, w_2)\right]$ has full rank.

Assumption A5: Firms' strategies are bounded from above, $\exists (\bar{p}_1, \bar{p}_2)$ such that $p_1 \leq \bar{p}_1$ and $p_2 \leq \bar{p}_2$. Both firms' prices are bounded below at 0.

Assumption A6: The support of w_1 and w_2 are subsets of $\mathbb{R}_+$.

Assumption A1 is a typical random sampling assumption. Assumption A2 specifies that econometric unobservables are uncorrelated with cost shifters. This is a standard exogeneity assumption. Notice that it explicitly rules out the special (and somewhat pathological) case where firm j's beliefs about its rival's price p_i depend on w in such a way as to exactly cancel out the effect of variation in w on p_j. Assumption A3 is standard as well. It is the typical rank condition which guarantees that there is enough exogenous variation in cost shifters $\mathbf{w}$ to identify the coefficients on prices in the demand specification. In Nash equilibrium, the reasoning used to support this is that variables that change marginal costs, $\mathbf{w}$, result in changes in prices through firms' equilibrium strategies. The same reasoning applies under the weaker rationality assumptions imposed here, since best-reply functions still depend on marginal costs. Assumption A4 is a rank condition that yields identification of the marginal cost parameters when the assumption of Nash equilibrium is imposed. Assumption A5 guarantees that the firms' action space is a lattice, which is necessary to ensure the game is supermodular. One can assume a fixed upper bound on the pricing space for simplicity, but more generally this could be a function of observables and model parameters endogenously determined in the model. We return to this issue in Section 3.3.1. Assumption A6 is motivated by the observation that factor prices, which are logical cost shifters, are necessarily non-negative. This assumption will facilitate the use of the covariance restrictions of assumption A2 in the construction of moment inequalities by preserving the direction of inequalities when multiplied by w_1 or w_2. If A6 does not hold, then additional care must be taken in the construction of moment inequalities from the bounds implied by k-level rational behavior. Alternatively, one could invoke a stronger conditional mean restriction on econometric unobservables with respect to exogenous covariates.

3.1 Identification of Demand

A1-A3 are sufficient conditions for point identification of the demand function parameters, regardless of whether firms play Nash equilibrium (which is equivalent to rationalizable strategies in this game by Milgrom and Roberts (1990), Theorem 5), or level-k rational strategies. This is because they uniquely solve the moment conditions

$$\begin{aligned} E\left[\left(Q_1 - \left(\alpha_1 - \beta_{11} p_1 + \beta_{12} p_2\right)\right) w_l\right] &= 0, \\ E\left[\left(Q_2 - \left(\alpha_2 - \beta_{22} p_2 + \beta_{21} p_1\right)\right) w_l\right] &= 0, \end{aligned} \tag{6}$$

$l = 0, 1, 2$, where $w_0 \equiv 1$.

3.2 Implications of Nash Equilibrium

If firms play Nash equilibrium, they play the solution to the first order conditions (4) and (5), which can be written as

$$\begin{aligned} mc_1 &= 2p_1 - \frac{1}{\beta_{11}}\left(\alpha_1 + \beta_{12} p_2 + u_1\right), \\ mc_2 &= 2p_2 - \frac{1}{\beta_{22}}\left(\alpha_2 + \beta_{21} p_1 + u_2\right). \end{aligned}$$

Combining this with the specification of the marginal cost functions, one can isolate the econometric unobservables ϵ_1, ϵ_2:

$$\begin{aligned} \epsilon_1 &= 2p_1 - \frac{1}{\beta_{11}}(\alpha_1 + \beta_{12}p_2 + u_1) - \mathbf{w}\boldsymbol{\gamma}_1, \\ \epsilon_2 &= 2p_2 - \frac{1}{\beta_{22}}(\alpha_2 + \beta_{21}p_1 + u_2) - \mathbf{w}\boldsymbol{\gamma}_2. \end{aligned} \tag{7}$$

The orthogonality conditions of assumption A2 yield a set of moment restrictions that identify the marginal cost parameters γ under A4. These moment conditions can be combined with those of (6) to consistently estimate model parameters by GMM.

3.3 Levels of Rationality

Now we suppose that instead of playing Nash Equilibrium, firms play level-k rational strategies. The moment conditions (6) derived from the demand specification continue to hold, so that the demand parameters, the α's and β's, remain identified. The equality restrictions (7) on marginal costs will no longer be satisfied, but can be replaced by moment inequality restrictions derived from level-k rational behavior. To illustrate this, we proceed to derive bounds on observed prices implied by k-level rational behavior following in the same vein as ALT, and next show how these bounds can be used to construct moment inequality restrictions. The implied moment inequalities can potentially then be used for estimation and inference on marginal cost parameters according to a variety of methods proposed in the recent literature, such as Andrews, Berry, and Jia (2004), Andrews and Guggenberger (2007), Beresteanu and Molinari (2008), Chernozhukov, Hong, and Tamer (2007), Galichon and Henry (2006), Pakes, Porter, Ho, and Ishii (2004), Romano and Shaikh (2006), and Rosen (2006).

First suppose that firms are only level-1 rational. Then they will only play undominated strategies. It can be shown that the set of undominated strategies for each firm j are $p_j \in \left[p_{j,1}^L, p_{j,1}^U\right]$ where

$$p_{j,1}^L = \max\left(p_j^*(0), mc_j\right), \; p_{j,1}^U = \min\left(\bar{p}_j, p_j^*(\bar{p}_{-j})\right). \tag{8}$$

The rationale for these lower and upper bounds is the following. Clearly, $p_j < mc_j$ is dominated, since that will yield negative profits. It remains to show that any $p_j < p_j^*(0)$ is also dominated. By definition, $p_j^*(0)$ is a best response to $p_{-j} = 0$. Since the best response functions are increasing in rival's price then indeed for *any* $p_{-j} > 0$, $p_j^*(0) < p_j^*(p_{-j})$. Because the profit function is strictly concave $p_j^*(0)$ will always give firm j higher profits than any $p_j < p_j^*(0)$, so that any such p_j is indeed dominated by $p_j^*(0)$. Similar reasoning gives that any $p_j > p_j^*(\bar{p}_{-j})$ is also dominated, so the upper bound on level-1 rational strategies is the minimum of $\bar{p}_j$ and $p_j^*(\bar{p}_{-j})$.

With level-1 rational strategies in hand, bounds on firms' prices for higher levels of rationality are defined recursively as

$$p_{j,k}^L = \max\left(p_j^*\left(p_{-j,k-1}^L\right), mc_j\right), \, p_{j,k}^U = \min\left(\bar{p}_j, p_j^*\left(p_{-j,k-1}^U\right)\right). \tag{9}$$

These bounds apply because a k-level rational firm will always play a best reply to a feasible $k-1$ level strategy for its opponent. This is precisely the reasoning of ALT (Claim 1) applied in the setting of the Bertrand pricing game. Since each best reply function is increasing in rival's strategy, the interval bounded by $p_j^*\left(p_{-j,k-1}^L\right)$ and $p_j^*\left(p_{-j,k-1}^U\right)$ is precisely the set of best replies to $k-1$ level rational strategies. Notice that Assumption A5 plays a key role in that it delivers a non-trivial upper bound on k-level rational behavior by bounding the effect of a rival's action on player j's payoff. In a non-linear model this could potentially be achieved without a bounded strategy space.

To see how these inequality restrictions on prices from k-level rationality have identifying power, notice that $p_{j,k}^L$ and $p_{j,k}^U$ are functions of model parameters, data, and econometric unobservables. Following ALT, this implies, for any level k, a set of moment inequality restrictions on model parameters. Below we illustrate the set of implied inequalities for the cases $k = 1$ and $k = 2$.

3.3.1 Moment Inequalities From Level-1 Rationality

Here we derive the inequality restrictions implied by k-level rational behavior for firm 1, for $k = 1$. The bounds for firm 2 have a symmetric derivation, and are thus omitted. Level 1 rationality implies that the observed price p_1 must lie within the interval of firm 1's undominated strategies, i.e. the bounds (8). The restriction that $p_{1,1}^{L} \leq p_1$ is equivalent to the two inequality restrictions

$$p_1^*(0) \leq p_{1,} \; mc_1 \leq p_1,$$

while the restriction from the upper bound, $p_1 \leq p_{1,1}^{U}$, is given by the two inequalities

$$p_1 \leq \bar{p}_1, \; p_1 \leq p_1^*(\bar{p}_2).$$

Multiplying these inequality restrictions by each component of $\mathbf{w}$, applying the definitions of $p_1^*(\cdot)$ and mc_1, and then taking expectations yields the following set of moment inequalities for $l = 0, 1, 2$, where $w_0 \equiv 1$,

$$\begin{aligned} E\left[w_l\left(2\beta_{11}p_1 - \alpha_1 + \beta_{11}\mathbf{w}\boldsymbol{\gamma}_1\right)\right] &\geq 0, \\ E\left[w_l\left(p_1 - \mathbf{w}\boldsymbol{\gamma}_1\right)\right] &\geq 0, \\ E\left[w_l\left(\bar{p}_1 - p_1\right)\right] &\geq 0, \\ E\left[w_l\left(\alpha_1 + \beta_{12}\bar{p}_2 + \beta_{11}\mathbf{w}\boldsymbol{\gamma}_1 - 2\beta_{11}p_1\right)\right] &\geq 0. \end{aligned}$$

The third inequality allows for the case where $\bar{p}_1$ is a function of observables and model parameters endogenously determined in the model. If $\bar{p}_1$ is assumed to be exogenously given, then it does not depend on model parameters, and this condition will not prove informative and therefore may be dropped.

Similar inequality restrictions for firm 2 can also be derived. These moment inequalities can then be used for estimation and inference.

3.3.2 Moment Inequalities From Level-2 Rationality

When $k = 2$ each firm will play best responses to some level-1 rational strategies of their opponent. Using the level k-rational bounds of (9), we have the following lower bounds for p_1:

$$p_1^* \left(p_{2,1}^L\right) \leq p_1, \; mc_1 \leq p_1,$$

where $p_{2,1}^L$ is the lower bound on firm 2's level-1 rational strategies. This is given by $p_{2,1}^L = \max\{p_2^*(0), mc_2\}$. Since $p_1^*(\cdot)$ is monotone, it follows that

$$p_1^* (p_2^* (0)) \leq p_1, \; p_1^* (mc_2) \leq p_1, \; mc_1 \leq p_1. \tag{10}$$

Upper bounds on p_1 from (9) are

$$p_1 \leq \bar{p}_1, \, p_1 \leq p_1^* \left(p_{2,1}^U\right).$$

Since $p_{2,1}^U = \min\{\bar{p}_2, p_2^*(\bar{p}_1)\}$, these restrictions are equivalent to

$$p_1 \leq \bar{p}_1, \, p_1 \leq p_1^* (\bar{p}_2), \, p_1 \leq p_1^* (p_2^* (\bar{p}_1)). \tag{11}$$

Multiplying both sides of all inequalities of (10) and (11) by each component of $\mathbf{w}$ and taking expectations gives for $l = 0, 1, 2$:

$$\begin{aligned}
E\left[w_l\left(4\beta_{22}\beta_{11}p_1 - 2\beta_{22}\left[\alpha_1 + \beta_{11}\mathbf{w}\boldsymbol{\gamma}_1\right] - \beta_{12}\left[\alpha_2 + \beta_{22}\mathbf{w}\boldsymbol{\gamma}_2\right]\right)\right] &\geq 0, \\
E\left[w_l\left(2\beta_{11}p_1 - \alpha_1 + \beta_{12}\mathbf{w}\boldsymbol{\gamma}_2 + \beta_{11}\mathbf{w}\boldsymbol{\gamma}_1\right)\right] &\geq 0, \\
E\left[w_l\left(p_1 - \mathbf{w}\boldsymbol{\gamma}_1\right)\right] &\geq 0, \\
E\left[w_l\left(\bar{p}_1 - p_1\right)\right] &\geq 0, \\
E\left[w_l\left(\alpha_1 + \beta_{12}\bar{p}_2 + \beta_{11}\mathbf{w}\boldsymbol{\gamma}_1 - 2\beta_{11}p_1\right)\right] &\geq 0, \\
E\left[w_l\left(2\beta_{22}\left[\alpha_1 + \beta_{11}\mathbf{w}\boldsymbol{\gamma}_1\right] + \beta_{12}\left[\alpha_2 + \beta_{21}\bar{p}_1 + \beta_{22}\mathbf{w}\boldsymbol{\gamma}_2\right] - 4\beta_{11}\beta_{22}p_1\right)\right] &\geq 0.
\end{aligned}$$

Level 2 rationality for firm 2 yields a symmetric set of moment inequalities. As was the case with level-1 rationality the moment inequalities can be used to provide a basis for estimation and inference.

These moment equalities appear to be more cumbersome notationally than those for level 1 rationality. For this reason, it is worth restating that the demand parameters, i.e. the α's and β's are identified separately from the moment equalities (6). The moment inequalities above are useful for their restrictions on the marginal cost parameters (the γ's), and the moment functions on the left-hand side of the inequalities are linear in these. Similar analysis could be applied to non-linear models as well, though the moment functions would of course no longer be linear.

3.4 Extension to Differentiated Products with Incomplete Information

In the previous section, we assumed that each firm j perfectly observes its rival's type, i.e. their rival's unobservable cost-shifter ϵ_{-j}. In practice, this might not be the case, resulting in a game of incomplete information. Our analysis can also be applied to the differentiated product model with incomplete information. By the same argument as in Section 2, the incomplete information version of this game is also supermodular. Each firm maximizes expected profit conditional on their own information set and their (rationalizable) beliefs. Suppose that each firm j observes ϵ_j but not ϵ_{-j}. Let $G(\epsilon_1, \epsilon_2)$ be the joint distribution of firm 1 and firm 2 types, and assume that this distribution is common knowledge. Then firm 1, for example, solves

$$\max_{p_1} \int \left[(p_1 - mc_1) \, Q_1(\mathbf{p}) \right] dG_1(\epsilon_2 | \epsilon_1),$$

where $G_1(\cdot|\epsilon_1)$ are firm 1's beliefs regarding firm 2's type. If firm 1 is level-1 rational, then $G_1(\cdot|\epsilon_1)$ has support on firm 2's strategy space. If firm 1 is level-2 rational, then $G_1(\cdot|\epsilon_1)$ has support within the set of firm 2's strategies that are level-1 rational, i.e. undominated, since firm 1 knows that firm 2 will not play a dominated strategy. If the firms are playing rationalizable strategies, then they are level-∞ rational (common knowledge of rationality), and $G_1(\cdot|\epsilon_1)$ has support on the set of firm 2's rationalizable strategies. In this case, due to Milgrom and Roberts' result, the extreme points of the set of rationalizable strategies are pure strategy BNE.

Maximizing expected profit, we obtain that the best replies are

$$p_1(\epsilon_1) = \frac{\alpha_1 + \beta_{12} E_{G_1}(p_2(\epsilon_2)|\epsilon_1) + u_1 + \beta_{11} mc_1}{2\beta_{11}},$$
$$p_2(\epsilon_2) = \frac{\alpha_2 + \beta_{21} E_{G_2}(p_1(\epsilon_1)|\epsilon_2) + u_2 + \beta_{22} mc_2}{2\beta_{22}}.$$

In some special cases, including those where types are independent, or where $G(\epsilon_1, \epsilon_2)$ is bivariate normal, the BNE is unique (Vives (1999, Chapter 8.1.2)). Hence, by Milgrom and Roberts (1990), Theorem 5, the set of rationalizable strategies is given by the unique BNE. The same analysis as in the previous Section can then be conducted, comparing the conclusions drawn under the assumption of Nash behavior versus level-k rationality.

Irrespective of whether the set of BNE is a singleton, the analysis of ALT applies. In particular, if firm j is level-1 rational, then $p_j \in \left[p^L_{j,1}, p^U_{j,1}\right]$ where $p^L_{j,1}, p^U_{j,1}$ are defined in equation (8). If firm j is level-2 rational, then $G_j(\cdot|\epsilon_j)$ assigns zero probability to $p_{-j} \notin \left[p^L_{-j,1}, p^U_{-j,1}\right]$. An iterative argument can be applied for level-k rationality.

4 Conclusion

In the familiar context of models of entry and auctions, ALT show how in empirical work the assumption of Nash Equilibrium can be weakened to rationalizability or even lower levels of rationality. Specifically, they show how these weaker solution concepts have identifying power that can be used for estimation and inference in these models. These findings are appealing in that the weaker solution concepts are direct implications of various levels of agents' rationality, whereas common knowledge of rationality is not sufficient to guarantee Nash behavior.

We have shown in this discussion how their insight can be fruitfully applied to the analysis of supermodular games. These games are useful for modeling settings with strategic complementarity, which are an important and empirically relevant class of economic models, well-studied in microeconomic theory. Specifically, Milgrom and Roberts (1990) and Vives (1990) have shown that in supermodular games, the largest and the smallest rationalizable strategies coincide with the largest and the smallest Nash Equilibria, respectively. This relationship between the two solution concepts implies that as one considers the implications of k-level rational behavior for increasing levels of k, the bounds on rational strategies will shrink to those of the Nash set. While the full sets of model predictions based on the two solution concepts may still differ when the equilibrium is not unique, this relationship between the two sets is useful for characterizing their relative identification power. For this reason, supermodular games provide appealing grounds in which to compare the identification power of various levels of rationality relative to that of an equilibrium assumption. In order to demonstrate the applicability of such analysis, we consider an oligopoly pricing game. We illustrate how the assumption of k-level rational behavior can be used to deliver moment equalities and inequalities, which can then be used for estimation and inference, where some parameters are possibly set-identified rather than point-identified.

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